Fall 2009 Math 245 Exam 2 Solutions

Exam scores: One quarter of the exam scores were below 70, one quarter between 70 and 75.5 (the median), one quarter between 75.5 and 80, and one quarter of the scores were above 80.

1. Carefully define the following terms:
   This problem tests the students' attention to detail and commitment to accurate definitions, which are very important in mathematics. A *constructive proof* of the existence of some object does so by explicitly finding the desired object. A *cardinal number* represents the size or cardinality of some set. The *symmetric difference* $A \Delta B$ for sets A, B is $(A - B) \cup (B - A)$ or $(A \cup B) - (A \cap B)$. The *power set* of a set S is the set consisting of all the subsets of S (including the empty subset). The *union* $A \cup B$ for sets A, B is the set $\{x : x \in A \text{ or } x \in B\}$.

2. Let $U = \{a, b, c, d\}, A = \{a, a, b, c\}, B = \{a, c\}, C = \{a, c, d\}$. Find $((A - B)\Delta C) \cap (C \cap B)^c$.
   This problem tests set operations. $A - B = \{b\}, ((A - B)\Delta C) = U, C \cap B = \{a, c\}, (C \cap B)^c = \{b, d\}, ((A - B)\Delta C) \cap (C \cap B)^c = \{b, d\}$.

3. Prove that if n is an even integer then $\lfloor \frac{n}{2} \rfloor = \frac{n}{2}$.
   This problem tests proofs with even numbers and floors. Because n is even, there is an integer k with $n = 2k$. Substituting, $\lfloor \frac{n}{2} \rfloor = \lfloor \frac{2k}{2} \rfloor = \lfloor k \rfloor = k = \frac{n}{2}$.

4. Let A, B be two sets. Prove that if $A \subseteq B$ then $B^c \subseteq A^c$.
   This problem tests proofs with subsets.

   SOLUTION 1: Direct proof. The hypothesis is that $A \subseteq B$. By definition of subset, this means that for all x, if $x \in A$ then $x \in B$. This is logically equivalent to its contrapositive, which is: for all x, if $x \notin B$, then $x \notin A$. But $(x \notin B) \equiv (x \in B^c)$, and $(x \notin A) \equiv (x \in A^c)$, so this implies: for all x, if $x \in B^c$, then $x \in A^c$. But this is the definition of $B^c \subseteq A^c$.

   SOLUTION 2: Proof by contradiction. Let $x \in B^c$, and suppose that $x \notin A^c$. Then $x \in A$, hence by hypothesis $x \in B$. But this contradicts $x \in B^c$, so our hypothesis (that $x \notin A^c$) was false, and $x \in A^c$. Hence we have proved that for all $x \in B^c$, $x \in A^c$; hence $B^c \subseteq A^c$.

5. Prove that, for all $n \in \mathbb{N}$, $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)^n = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$.
   This problem tests matrix multiplication and proof by induction. Call

the predicate $S(n)$. The base case is $S(1)$, i.e. $n = 1$, which is that $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)^1 = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$, true. We now suppose $S(n)$ is true and try to prove $S(n+1)$. $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)^{n+1} = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)^n \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right) = \left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right) = \left(\begin{smallmatrix}1 & n+1\\ 0 & 1\end{smallmatrix}\right)$, where we used the definition of exponentiation, our inductive hypothesis, and matrix multiplication respectively. Comparing the first and last expressions proves $S(n+1)$, as desired.

6. Use the Euclidean algorithm to calculate $\gcd(605, 847)$.
   This problem tests the Euclidean algorithm. $847 = 1 \cdot 605 + 242$. $605 = 2 \cdot 242 + 121$. $242 = 2 \cdot 121 + 0$. Hence $\gcd(605, 847) = 121$.

7. For all odd integers n, prove that n^3 is odd.
   This problem tests the definition of odd. Our hypothesis is that n is odd; by definition of odd there is some integer k with $n = 2k + 1$. $n^3 = (2k+1)^3 = (2k)^3 + 3(2k)^2 + 3(2k) + 1 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1 = 2s + 1$, for integer $s = 4k^3 + 8k^2 + 4k$. Hence by definition of odd, n^3 is odd.

8. Prove that $\sqrt{3}$ is irrational.
   This problem tests proofs by contradiction. Suppose that $\sqrt{3} = \frac{m}{n}$, where m, n have no common factors. Squaring, we get $3 = \frac{m^2}{n^2}$, hence $m^2 = 3n^2$. So $3|m \cdot m$; but 3 is prime, so $3|m$. Write $m = 3k$, and substitute into $m^2 = 3n^2$ to get $9k^2 = 3n^2$ or $3k^2 = n^2$. So $3|n \cdot n$; but 3 is prime, so $3|n$. So 3 is a common factor of both m, n, which contradicts our hypothesis that m, n have no common factors.

9. Prove that $x^2 + 2x < 8$ if and only if $|x + 1| < 3$.
   This problem tests proofs of biconditional theorems. It is important to prove both directions; this can be done by proving each direction separately, or by being very careful. $x^2 + 2x < 8$, iff $x^2 + 2x - 8 < 0$, iff $(x + 4)(x - 2) < 0$, iff exactly one of $(x + 4), (x - 2)$ is negative, iff $x + 4 > 0$ and $x - 2 < 0$, iff $-4 < x < 2$, iff $-3 < x + 1 < 3$, iff $|x + 1| < 3$.

10. Consider the two-element Boolean algebra $\{0, 1\}$. Prove the absorption theorem: $\forall a \forall b, a \oplus (a \odot b) \equiv a$.
    This problem tests understanding of Boolean algebra arithmetic. Fortunately this Boolean algebra is small, so there are only four cases to test.

a	b	$a \odot b$	$a \oplus (a \odot b)$
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

Comparing the first and last column proves the theorem.